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METHODS OF GROUP-THEORETIC ANALYSIS IN THE PROBLEM OF THE SLIDING MOTION OF A SPREADING THIN FILM OF NON-LINEARLY VISCOUS LIQUID†

S. L. TONKONOG, V. A. CHUGUNOV and L. D. ESKIN

Kazan

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Asymptotic formulae are established for the dynamics of the free surface of a thin film, and the law of motion of the boundary determined, for a layer of non-linearly viscous liquid spreading over a horizontal base with sliding. Computational results are presented.

1. STATEMENT OF THE PROBLEM

The dynamics of flow of non-linearly viscous liquid films obeying a power constitutive law are described in the one-dimensional approximation by the following equation [1, 2]

$$\partial l / \partial t = \partial q^\varepsilon / \partial x, \quad t > 0, \quad 0 < x < x_0 \tag{1.1}$$

where $l(x, t)$ is the film thickness, and q^ε is the flux of the liquid

$$q^\varepsilon = \operatorname{sign}\left(\frac{\partial l}{\partial x}\right)\left(\frac{l^{2+n}}{n+2}\left|\frac{\partial l}{\partial x}\right|^n + \varepsilon l^{1+m}\left|\frac{\partial l}{\partial x}\right|^m\right), \quad n > m \tag{1.2}$$

where the constant n is determined by the constitutive law, and ε and m are determined by the law governing the sliding of the film relative to the base, which is assumed here to be flat.

Note that in applications, after non-dimensionalization, it usually turns out that $\varepsilon < 1$ [1]. We shall here assume this to be the case.

Equation (1.1) will be considered with the boundary conditions

$$x = 0: \partial l / \partial x = 0 \text{ (consequently, } q^\varepsilon = 0) \tag{1.3}$$

$$x = x_0(t) \text{ (point of the front) } l = 0, \ q^\varepsilon = 0 \tag{1.4}$$

$$t = 0: l = M\delta(x) \tag{1.5}$$

where M is the mass of liquid per unit width of the film, and $\delta(x)$ is the delta-function. (The boundary conditions (1.3) and (1.4) ensure that the solution l can be extended as an even function into the domain $x < 0$, preserving smoothness at $x = 0$.)

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The mixed problem (1.1)–(1.5) describes the spreading of a non-Newtonian liquid concentrated at the starting time along the axis $x=0$. The absence of sinks at $x=0$ and $x=x_0(t)$ guarantees conservation of the mass of the spreading film, so that

$$\int_0^{x_0(t)} l(x,t)dx = \frac{1}{2}M \tag{1.6}$$

Together with (1.1), we shall also consider the unperturbed equation

$$\partial l / \partial t = \partial q^0 / \partial x \tag{1.7}$$

where the flux q^0 is obtained from (1.2) by putting $\varepsilon = 0$.

It has been shown [2] that the unperturbed equation (1.7) is invariant under the group of dilatations with infinitesimal operator

$$X_\lambda = t(1+\lambda(n+1))\frac{\partial}{\partial t} + (2+\lambda)x\frac{\partial}{\partial x} + l\frac{\partial}{\partial l} \tag{1.8}$$

and moreover only a self-similar solution of Eq. (1.7), which is invariant under the group with operator (1.8) with $\lambda=-3$, satisfies conditions (1.3)–(1.5) or, consequently, the law of conservation of mass. This solution of Eq. (1.7) may be determined in closed form [2]

$$l(x,t) = D_n t^{-\alpha}(\xi_0^{1+1/n} - \xi^{1+1/n})^\beta, \quad 0 < \xi < \xi_0 \tag{1.9}$$

$$\xi = xt^{-\alpha}, \quad D_n = (\alpha^{1/n}\gamma^{-1})^\beta, \quad x_0(t) = \xi_0 t^\alpha$$

$$\alpha = \frac{1}{3n+2}, \quad \beta = \frac{n}{2n+1}, \quad \gamma = \frac{n+1}{2n+1}$$

The constant ξ_0 is determined from the mass-conservation condition (1.6).

However, the introduction of sliding violates the symmetry of (1.8). The perturbed equation (1.1) is not invariant under the group of dilatations with operator X_λ; as shown in [2], it admits of a dilatation group with the operator

$$X = (m-2n-1)t\frac{\partial}{\partial t} + (2m-2n-1)x\frac{\partial}{\partial x} + (m-n)\frac{\partial}{\partial l} \tag{1.10}$$

Unfortunately, a solution of Eq. (1.1) which is invariant under this group cannot satisfy the conservation law (1.6); hence it cannot be a solution of the problem of free spreading of the film as represented by (1.1)–(1.5).

Below we shall develop an algorithm for solving problem (1.1)–(1.5), based on constructing an asymptotic expansion ($\varepsilon < 1$) of a solution, invariant under the group of dilatations admissible by Eq. (1.1) and transforming not only t, x and l but also the parameter ε. Such groups were used in [3] to investigate the non-linear wave equation.

2. CONSTRUCTION OF THE TRANSFORMATION GROUP

The infinitesimal operator of the dilatation group operating on the variables t, x, l and the parameter ε and leaving Eq. (1.1) invariant will be sought in the form (generalization of (1.8) for $\lambda=-3$))

$$Y = -(3n+2)t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + l\frac{\partial}{\partial l} + s\varepsilon\frac{\partial}{\partial \varepsilon}$$

where the constant s is yet to be determined. The operator Y generates a dilatation group

$$\tilde{t} = \omega^{-(3n+2)}t, \quad \tilde{x} = \omega^{-1}x, \quad \tilde{l} = \omega l, \quad \tilde{\varepsilon} = \omega^{s}\varepsilon \tag{2.1}$$

The presumed invariance of Eq. (1.1) under the group (2.1) implies that $s = 3(n-m)+1$

$$Y = -(3n+2)t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + l\frac{\partial}{\partial l} + (3(n-m)+1)\varepsilon\frac{\partial}{\partial \varepsilon} \tag{2.2}$$

after which we obtain three independent group invariants

$$I_1 = \xi = xt^{-\alpha}, \quad I_2 = \eta = \varepsilon t^{\alpha(1+3(n-m))}, \quad I_3 = lt^{\alpha}$$

Using the standard invariant form of the solution of Eq. (1.1) as $I_3 = \psi(\xi, \eta)$, we obtain

$$l = t^{-\alpha}\psi(\xi, \eta) \tag{2.3}$$

Assuming that l is a monotone decreasing function of x and substituting (2.3) into (1.1), we obtain an equation for the unknown function ψ

$$\alpha[\psi + \xi\psi_{\xi} - (1+3(n-m))\eta\psi_{\eta}] = \frac{\partial}{\partial \xi}\left(\frac{\psi^{n+2}(-\psi_{\xi})^{n}}{n+1} + \eta\psi^{m+1}(-\psi_{\xi})^{m}\right) \tag{2.4}$$

The law of motion of the boundary $x_0 = x_0(t)$ in the new variables ξ, η may be written in the form $\xi_0 = g(\eta)$ where $g(\eta)$ is a new unknown function, to be determined together with ψ. When solving boundary-value problems it is more convenient to work with fixed boundaries. We shall therefore change from ξ to a new variable: $z = \xi / g(\eta)$. Transforming Eq. (2.4) to the variables z, η and keeping the old notation $\psi(z, \eta)$ for the unknown function, we deduce the following equation from (2.4)

$$\alpha\left[\psi + z\psi_{z} - (1+3(n-m))\eta\left(\psi_{\eta} - \frac{z\psi_{z}g_{\eta}}{g}\right)\right] = \frac{\partial q}{\partial z} \tag{2.5}$$

The law of conservation of mass (1.6) implies the following relationship for the function $\psi(z, \eta)$

$$g(\eta)\int_0^1 \psi(z, \eta)dz = \frac{1}{2}M \tag{2.6}$$

while the boundary conditions (1.3) and (1.4) become

$$z = 0: \partial\psi / \partial z = 0; \quad z = 1: \psi = 0, \quad q = 0 \tag{2.7}$$

Here and in (2.5) the flux q is given by the following formula

$$q = \frac{\psi^{n+2}(-\psi_{z})^{n}}{(n+2)g^{n+1}} + \eta\frac{\psi^{m+1}(-\psi_{z})^{m}}{g^{m+1}} \tag{2.8}$$

3. SOLUTION OF THE PROBLEM

A solution $\psi(z, \eta)$ of the boundary-value problem (2.5), (2.7) will be sought as a series in powers of the invariant η; the unknown function $g(\eta)$ will also be sought in this form. Confining ourselves to terms of order less than $O(\eta^2)$, we write

$$\psi(z,\eta) = \upsilon(z) + \eta u(z) + O(\eta^2), \quad g(\eta) = a + b\eta + O(\eta^2) \tag{3.1}$$

with unknown functions $\upsilon(z)$ and $u(z)$ and constant (independent of z) coefficients a and b. Substituting (3.1) into Eq. (2.5), after first multiplying it by $g^{n+1}(\eta)$, and comparing the coefficients of η^0, η in the resulting equality, we obtain equations for the unknown functions υ and u

$$\alpha a^{n+1}(\upsilon + z d\upsilon / dz) = dq_0 / dz \tag{3.2}$$

$$\alpha[a^{n+1}(zu)_z + (n+1)a^n b(z\upsilon)_z - (1+3(n-m))(a^{n+1}u - a^n bz\upsilon_z)] = dq_1 / dz \tag{3.3}$$

where

$$q_0 = \upsilon^{n+2}(-\upsilon_z)^n / (n+2) \tag{3.4}$$

$$q_1 = \upsilon^{m+1}(-\upsilon_z)^m a^{n-m} + [(n+2)\upsilon^{n+1}u(-\upsilon_z)^n - n\upsilon^{n+2}(-\upsilon_z)^{n-1}u_z] / (n+2) \tag{3.5}$$

Substituting (3.1) into the boundary conditions (2.7) and (2.8) and expanding in powers of η, we get the following conditions

$$z = 0: \upsilon_z = 0; \quad z = 1: \upsilon = 0, \quad q_0 = 0 \tag{3.6}$$

for Eq. (3.2) and

$$z = 0: u_z = 0; \quad z = 1: u = 0, \quad q_1 = 0 \tag{3.7}$$

for Eq. (3.3).

Finally, the law of conservation of mass (2.6) implies

$$\int_0^1 \upsilon(z)dz = \frac{1}{2a}M, \quad \int_0^1 \left(\upsilon + \frac{a}{b}u\right)dz = 0 \tag{3.8}$$

the second equality here being a direct result of (3.3) and the boundary conditions (3.6) and (3.7)—this may be verified by integrating with respect to z from 0 to 1 on both sides of Eq. (3.3).

Equation (3.2) is a non-linear equation in $\upsilon(z)$. Its order may be reduced and the unique solution that satisfies the boundary conditions (3.6) determined in closed form

$$\upsilon(z) = C_n a^\gamma (1 - z^{1+1/n})^\beta, \quad C_n = [\gamma^{-1}((n+2)\alpha)^{1/n}]^\beta \tag{3.9}$$

The unknown constant a is found from the first condition of (3.8), which, after substituting (3.9), becomes an equation for a. Finally, we obtain

$$a = \left[\frac{M(n+1)}{2nC_n}\left(B\left(\frac{3n+1}{n+1}, \beta\right)\right)^{-1}\right]^{(n+1)\alpha/\gamma} \tag{3.10}$$

(B is the Euler beta-function).

We have thus determined the zero-order terms in the expansions (3.1). We will now determine the unknown function and the coefficient b.

Equation (3.3) is a second-order linear inhomogeneous differential equation in u, with coefficients expressed in terms of the function υ and its derivatives of order up to two inclusive and the constants a and b. The other coefficients of the expansion of $\psi(z,\eta)$ in powers of η are solutions of second-order linear equations, as is the case for u. We introduce a new unknown function $w=u+ba^{-1}\upsilon$, thus giving Eq. (3.3) the form

$$\alpha(1+3(n-m))a^{n+1}w=\frac{d}{dz}(q_1-\alpha zua^{n+1}-\alpha(4n-3m+2)a^n bz\upsilon) \tag{3.11}$$

Substituting q_1 as given by (3.5) into the right-hand side of (3.11), replacing u by $w-ba^{-1}\upsilon$, and using (3.2) and (3.4), we get an equation for w

$$-\alpha(1+3(n-m))a^{n+1}w=\frac{d}{dz}\left[\upsilon^{n+1}(-\upsilon_z)^n w-\alpha a^{n+1}zw-\frac{n}{n+2}\upsilon^{n+2}(-\upsilon_z)^{n-1}w_z\right]+f$$

$$f=\frac{d}{dz}[\upsilon^{m+1}(-\upsilon_z)^m a^{n-m}-3\alpha(2n-m+1)a^n bz\upsilon] \tag{3.12}$$

Conditions (3.6) and (3.7) imply boundary conditions

$$z=0: w_z=0;\quad z=1: w=0 \tag{3.13}$$

for the function w. The coefficients of Eq. (3.12) can be calculated using (3.9). The result is

$$\upsilon^{n+2}(-\upsilon_z)^{n-1}=(n+2)\alpha\gamma^{-1}a^{n+1}z^{1-1/n}(1-z^{1+1/n}),\quad \upsilon^{n+1}(-\upsilon_z)^n=(n+2)\alpha a^{n+1}z$$
$$\upsilon^{n+2}(-\upsilon_z)^n=(n+2)\alpha C_n a^{2\gamma}z(1-z^{1+1/n})^\beta \tag{3.14}$$
$$\upsilon^{m+1}(-\upsilon_z)^m=(C_n a^\gamma)^{2m+1}\gamma^m z^{m/n}(1-z^{1+1/n})^\delta,\quad \delta=\frac{n-m}{2n+1}$$

It follows from these formulae that Eq. (3.12) may be rewritten as

$$(3(n-m)+1)w=\frac{d}{dz}[n\gamma^{-1}z^{1-1/n}(1-z^{1+1/n})w_z-(n+1)zw]-\frac{f(z)}{\alpha a^{n+1}} \tag{3.15}$$

where the coefficients have only an algebraic singularity (a branch-point) at $z=0$ and $z=1$ and can be expanded in the neighbourhood of $z=1$ as power series of the form

$$(z-1)^\mu\sum_{k=1}^{\infty}a_k(z-1)^k \tag{3.16}$$

We will now need higher-order terms of the expansions

$$z^{1-1/n}(1-z^{1+1/n})\sim\frac{n+1}{n}(1-z),$$
$$\upsilon^{m+1}(-\upsilon_z)^m\sim(C_n a^\gamma)^{2m+1}\gamma^m\left(\frac{n+1}{n}\right)^\delta(1-z)^\delta \tag{3.17}$$

$$\upsilon^{n+2}(-\upsilon_z)^n\sim\beta^{-\beta}((n+2)\alpha a^{n+1})^{2\gamma}(1-z)^\beta,\quad \upsilon\sim\beta^{-\beta}((n+2)\alpha a^{n+1})^{\beta/n}(1-z)^\beta$$

The general solution of the inhomogeneous equation (3.16) is

$$w = Aw_1 + Bw_2 + w_3 \tag{3.18}$$

where $w_{1,2}$ are any of two independent solutions of the homogeneous equation

$$(3(n-m)+1)w = \frac{d}{dz}[n\gamma^{-1} z^{1-1/n}(1-z^{1+1/n})w_z - (n+1)zw] \tag{3.19}$$

A and B are arbitrary constants, and w_3 is any particular solution of Eq. (3.15).

By well-known results of the analytic theory of differential equations, the solutions w_1, w_2 and w_3 may also be sought as expansions in powers of $z-1$, similar to the expansions (3.16) for the coefficients of the equation. Defining

$$w = (1-z)^{\tau}(1+b_1(1-z)+\ldots) \tag{3.20}$$

substituting this expansion into (3.19) and using the first relationship of (3.17), we obtain the characteristic equation in τ

$$(2n+1)\tau^2 + (n+1)\tau = 0$$

whence we obtain two independent solutions of Eq. (3.19), which admit of the following power series expansions in the neighbourhood of $z=1$

$$w_1 = (1-z)^{-\gamma}(1+b_{11}(1-z)+\ldots), \quad w_2 = 1+b_{21}(1-z)+\ldots \tag{3.21}$$

A particular solution of the inhomogeneous equation (3.15) may also be sought as a power series

$$w_3 = (1-z)^{\zeta}(c_1 + c_2(1-z)+\ldots) \tag{3.22}$$

It follows from (3.17) that the leading term in the expansion of the free term of Eq. (3.15) is

$$\delta\alpha^{-1}\gamma^m\left(\frac{n+1}{n}\right)^{\delta} C_n^{2m+1} a^{2\gamma(m-n)}(1-z)^{\delta-1} \tag{3.23}$$

Substituting the expansion (3.22) into Eq. (3.15), we obtain the exponent ζ and the coefficient c_1; the other coefficients c_i; $i \geq 2$, will not be needed. Using (3.17) and (3.23), we obtain

$$\zeta = \delta > 0, \quad c_1 = -\frac{2n+1}{\alpha(2n+1-m)}\left(\frac{n+1}{n}\right)^{\delta} C_n^{2m+1} a^{-\delta} \tag{3.24}$$

Since $\zeta > 0$, it follows from (3.21) and (3.22) that the boundary condition (3.13) for the solution w of Eq. (3.15) at $z=1$ determined by (3.18) will be satisfied only if we put $A=B=0$ in (3.18).

We shall now verify that the function $u = w_3 - ba^{-1}\upsilon$ satisfies the second boundary condition at $z=1$ that is, the condition $q_1 = 0$ in (3.7). Noting that $u=0$ at $z=1$, while $n>m$, we verify with the help of (3.14) that the first two terms in expression (3.5) for q_1 vanish at $z=1$. It remains to check that the last term in (3.5), i.e. $\upsilon^{n+2}(-\upsilon_z)^{n-1}u_z$, also vanishes at $z=1$. We have

$$\upsilon^{n+2}(-\upsilon_z)^{n-1}u_z = \upsilon^{n+2}(-\upsilon_z)^{n-1}(w_3)_z + ba^{-1}\upsilon^{n+2}(-\upsilon_z)^n \tag{3.25}$$

The second term on the right-hand side of (3.25) vanishes at $z=1$ by (3.14), while the first may be expanded in the neighbourhood of $z=1$ as a series in powers of $1-z$ whose leading term is equal to $d(1-z)^\zeta$ (where d is an unimportant constant); this is readily verified using (3.14), (3.17) and (3.22). Since $\zeta>0$, the first term on the right-hand side of (3.25) also vanishes at $z=1$.

The solution $w=w_3$ of Eq. (3.18) (and together with it the function u) depends on the coefficient b in (3.1). This coefficient is uniquely defined by the condition $w_z=0$ at $z=0$.

We now observe that as $t\to 0$ the point $x_0(t)=t^a g(\eta)$ on the front tends to zero, i.e. the support of the solution we have constructed contracts to the origin. Hence it follows that our solution, which satisfies the law of conservation (1.6), will automatically satisfy the initial condition (1.5) also.

We will now indicate a simple and convenient way of calculating the coefficient b in the expansion (3.1) of $g(\eta)$ (it is this coefficient, together with the parameter a, that defines the effect of sliding apart from $O(\eta^2)$). To that end, we note, first, that $f(z)$, the free term of Eq. (3.15), may be written in the form

$$f(z)=f_1(z)+bf_2(z)$$
$$f_1=a^{n-m}(\upsilon^{m+1}(-\upsilon_z)^m)_z,\quad f_2=-3a^n\alpha(2n-m+1)(z\upsilon)_z$$

The functions $f_{1,2}$ and the coefficients of the homogeneous equation (3.19) are independent of b. The linearity of Eq. (3.15) implies that its solution w_3 may be written

$$w_3=w_{31}(z)+bw_{32}(z) \tag{3.26}$$

where w_{31} and w_{32} are solutions of Eq. (3.15) with $f(z)$ replaced by $f_1(z)$ and $f_2(z)$, respectively. These solutions no longer depend on b, and it follows from (3.26) and the boundary condition $(w_3)_z=0$ at $z=0$ that

$$b=-(w_{31})_z/(w_{32})_z \tag{3.27}$$

Since the leading term of the expansion of $f_1(z)$ in powers of $1-z$ is the same as that of $f(z)$, the leading terms in the expansions of w_3 and w_{31} are also the same, i.e.

$$w_{31}\sim C_1(1-z)^\zeta,\quad z\to 1 \tag{3.28}$$

(ζ and c_1 are determined by (3.24)). The derivative is

$$(w_{31})_z\sim-\zeta c_1(1-z)^{\zeta-1},\quad z\to 1 \tag{3.29}$$

The leading term of the expansion of f_2 in powers of $1-z$ may be found using the expression from (3.17) for υ; it is

$$f_2\sim c_2(1-z)^{-\gamma},\quad c_2=3(\gamma+\delta)\gamma^{-\beta}((n+2)\alpha)^{\beta/n}a^{-\beta}$$

We can now determine the leading terms of the expansions of w_{32} and its derivative $(w_{32})_z$ in powers of $1-z$

$$w_{32}\sim\frac{c}{n}(1-z)^\beta,\quad (w_{32})_z\sim-\frac{c}{2n+1}(1-z)^{-\gamma}$$
$$c=-3(2n-m+1)\beta^\gamma((n+2)\alpha)^{\beta/n}a^{-\beta} \tag{3.30}$$

The asymptotic expansions (3.28)–(3.30) enable one approximately to determine the initial data of the Cauchy problem at a point $z=1-\varepsilon$ (where ε is sufficiently small), so that numerical methods can be used to find the solutions w_{31} and w_{32} and their derivatives at $z=0$ to any desired accuracy, and then, using

(3.27) to determine the coefficients b.

The non-monotone nature of the function $b=b(m)$ (the curve has a single minimum) is easily explained if one considers that in the sliding model used here (which is indeed commonly adopted today, see [4]) the non-dimensional velocity of sliding is equal to $\upsilon_0=\varepsilon\sigma^m$ (where $\sigma=/|\partial/\partial x|$ is the shear stress). The magnitude of σ is clearly significant in the boundary zone, near a point of the front $x=x_0(t)$, since $\partial/\partial x\to\infty$ as $x\to x_0(t)$, and conversely $\partial/\partial x\to 0$ as $x\to 0$. Consequently, increasing m sharply increases the velocity of sliding in the boundary zone, implying a higher coefficient b. Reducing m, on the one hand, reduces the velocity of sliding in the boundary zone but, on the other, it extends the region in which sliding is significant. When the second factor predominates over the first, the parameter b again begins to increase.

In conclusion, we note a simple power dependence of b on a, i.e. on the mass of the film and the parameter n of the constitutive law. Indeed, it follows from (3.9) and (3.14) that

$$f_1(a)=f_1(1)a^{\varphi},\quad f_2(a)=f_2(1)a^{\chi}$$

$$\varphi=\frac{2n^2+2n+m+1}{2n+1},\quad \chi=\frac{2n^2+2n+1}{2n+1}$$

Obviously, an analogous relationship links the solutions w_{31}, w_{32} and their derivatives with respect to z, on the one hand, and the parameter a. Hence, by (3.27), we obtain

$$b(a)=b(1)a^{m/(m+1)}$$

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